On the generalized higher-order q-Bernoulli numbers and polynomials

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Abstract : In this paper we give some interesting equation of p-adic q-integrals on $\mathbb{Z}_p$. From those p-adic q-integrals, we present a systemic study of some families of extended Carlitz q-Bernoulli numbers and polynomials in p-adic number field.

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80

Key words : Bernoulli numbers, Bernoulli polynomials, generalized higher-order q-Bernoulli numbers and polynomials

1. Introduction

Let p be a fixed prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integer, the field of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$.

Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we normally assume $|q| < 1$, and if $q \in \mathbb{C}_p$, we normally assume $|1-q|_p < 1$. We use the notation

$$[x]_q = \frac{1-q^x}{1-q}.$$

The q-factorial is defined as

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$$

and the Gaussian q-binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q!}, \text{ (see [9]).} \tag{1}$$

Note that

$$\lim_{q\to 1} \binom{n}{k}_q = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

From (1), we easily see that

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q = q^{n-k}\binom{n}{k-1}_q + \binom{n}{k}_q, \text{ (see [8, 11]).}$$

For a fixed positive integer f, $(f,p) = 1$, let

$$X = X_f = \varprojlim_N (\mathbb{Z}/fp^N\mathbb{Z}), \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{\substack{0<a<fp \\ (a,p)=1}} (a + fp\mathbb{Z}_p), \text{ and}$$

$$a + fp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{fp^N}\},$$

where $a \in \mathbb{Z}$ and $0 \leq a < fp^N$(see [1-14]).

We say that f is a uniformly differential function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$ if the difference quotients

$$F_f(x,y) = \frac{f(x) - f(y)}{x - y}$$

have a limit $l = f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{Z}_p)$, let us begin with the expression

$$\frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x = \sum_{0 \leq x < p^N} f(x)\mu_q(x + p^N\mathbb{Z}_p),$$

representing a q-analogue of the Riemann sums for f, (see [8-18]). The integral of f on $\mathbb{Z}_p$ is defined as the limit $(N \to \infty)$ of the sums (if exists). The p-adic q-integral (or q-Volkenborn integrals of $f \in UD(\mathbb{Z}_p)$) is defined by

$$I_q(f) = \int_X f(x)d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{0 \leq x < p^N} f(x)q^x, \text{ (see [12]).} \tag{2}$$

Carlitz's q-Bernoull numbers $\beta_{k,q}$ can be defined recursively by $\beta_{0,q} = 1$ and by the rule that

$$q\left(q\beta + 1\right)^k - \beta_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing β^i by $\beta_{i,q}$, (see [1-13]).

It is well known that

$$\beta_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) = \int_X [x]_q^n d\mu_q(x), n \in \mathbb{Z}_+, \text{ (see [9]),}$$

and

$$\beta_{n,q}(x) = \int_{\mathbb{Z}_p} [y + x]_q^n d\mu_q(y) = \int_X [y + x]_q^n d\mu_q(y), n \in \mathbb{Z}_+,$$

where $\beta_{n,q}(x)$ are called the n-th Carlitz's q-Bernoulli polynomials (see [9, 12, 13]).

Let χ be the Dirichlet's character with conductor $f \in \mathbb{N}$. Then the generalized Carlitz's q-Bernoulli numbers attached to χ are defined as follows:

$$\beta_{n,\chi,q} = \int_X \chi(x)[x]_q^n d\mu_q(x), \text{ (see [13]).}$$

Recently, many authors have studied in the different several areas related to q-theory (see [1-13]). In this paper we present a systemic study of some families of multiple Carlitz's q-Bernoulli numbers and polynomials by using the integral equations of p-adic q-integrals on $\mathbb{Z}_p$. First, we derive some interesting the equations of p-adic q-integrals on $\mathbb{Z}_p$. From these equations, we give some interesting formulae for the higher-order Carlitz's q-Bernoulli numbers and polynomials in the p-adic number field.

2. On the generalized higher-order q-Bernoulli numbers and polynomials

In this section we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. We first consider the q-extension of Bernoulli polynomials as follows:

$$\sum_{n=0}^{\infty}\beta_{n,q}(x)\frac{t^n}{n!}=\int_{\mathbb{Z}_p}q^{-y}e^{[x+y]_q t}d\mu_q(t)=-t\sum_{m=0}^{\infty}e^{[x+m]_q t}q^{x+m}. \tag{3}$$

From (3), we note that

$$\begin{aligned}
\beta_{n,q}(x)&=\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-q^x)^l\frac{l}{[l]_q}\\
&=\frac{1}{(1-q)^{n-1}}\sum_{l=0}^{n}\binom{n}{l}(-q^x)^l\left(\frac{l}{1-q^l}\right)\\
&=\frac{n}{(1-q)^{n-1}}\sum_{l=0}^{n-1}\binom{n-1}{l}q^{(l+1)x}\left(\frac{1}{1-q^{l+1}}\right)(-1)^{l+1} \qquad (4)\\
&=\frac{-n}{(1-q)^{n-1}}\sum_{m=0}^{\infty}q^{m+x}\sum_{l=0}^{n-1}\binom{n-1}{l}q^{l(x+m)}\\
&=-n\sum_{m=0}^{\infty}q^{m+x}[x+m]_q^{n-1}.
\end{aligned}$$

Note that

$$\lim_{q\to 1}\beta_{n,q}(x)=-n\sum_{m=0}^{\infty}(x+m)^{n-1}=B_n(x),$$

where $B_n(x)$ are called the n-th ordinary Bernoulli polynomials. In the special case, $x=0$, $\beta_{n,q}(0)=\beta_{n,q}$ are called the n-th q-Bernoulli numbers.

By (4), we have the following lemma.

Lemma 1. For $n\geq 0$, we have

$$\begin{aligned}
\beta_{n,q}(x)&=\int_{\mathbb{Z}_p}q^{-y}[x+y]_q^n d\mu_q(y)=-n\sum_{m=0}^{\infty}q^{m+x}[x+m]_q^{n-1}\\
&=\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-q^x)^l\frac{l}{[l]_q}.
\end{aligned}$$

Now, we consider the q-Bernoulli polynomials of order $r\in\mathbb{N}$ as follows:

$$\sum_{n=0}^{\infty}\beta_{n,q}^{(r)}(x)\frac{t^n}{n!}=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}q^{-(x_1+\cdots+x_r)}e^{[x+x_1+\cdots+x_r]_q t}d\mu_q(x_1)\cdots d\mu_q(x_r). \tag{5}$$

By (5), we see that

$$\begin{aligned}
\beta_{n,q}^{(r)}(x)&=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}q^{-(x_1+\cdots+x_r)}[x+x_1+\cdots+x_r]_q^n d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}\left(\frac{l}{[l]_q}\right)^r.
\end{aligned}$$

In the special case, $x = 0$, the sequence $\beta_{n,q}^{(r)}(0) = \beta_{n,q}^{(r)}$ is refereed as the q-extension of Bernoulli numbers of order r. For $f \in \mathbb{N}$, we have

$$\begin{aligned}\beta_{n,q}^{(r)}(x) &= \underbrace{\int_X \cdots \int_X}_{r \text{ times}} q^{-(x_1+\cdots+x_r)}[x+x_1+\cdots+x_r]_q^n d\mu_q(x_1)\cdots d\mu_q(x_r) \\ &= \frac{1}{(1-q)^n}\sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{l(x+a_1+\cdots+a_r)} l^r}{[lf]_q^r} \\ &= [f]_q^{n-r} \sum_{a_1,\cdots,a_r=0}^{f-1} \beta_{n,q^f}^{(r)}\left(\frac{a_1+\cdots+a_r+x}{f}\right).\end{aligned} \tag{6}$$

By (5) and (6), we obtain the following theorem.

Theorem 2. For $r \in \mathbb{Z}_+, f \in \mathbb{N}$, we have

$$\begin{aligned}\beta_{n,q}^{(r)}(x) &= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\sum_{a_1,\cdots,a_r=0}^{f-1}\binom{n}{l}(-1)^l q^{l(a_1+\cdots+a_r+x)}\frac{l^r}{[lf]_q^r} \\ &= [f]_q^{n-r}\sum_{a_1,\cdots,a_r=0}^{f-1}\beta_{n,q^f}^{(r)}\left(\frac{a_1+\cdots+a_r+x}{f}\right).\end{aligned}$$

Let χ be the primitive Dirichlet's character with conductor $f \in \mathbb{N}$. Then the generalized q-Bernoulli polynomials attached to χ are defined by

$$\sum_{n=0}^{\infty}\beta_{n,\chi,q}(x)\frac{t^n}{n!} = \int_X \chi(y)q^{-y}e^{[x+y]_q t}d\mu_q(y). \tag{7}$$

From (7), we derive

$$\begin{aligned}\beta_{n,\chi,q}(x) &= \int_X \chi(y)q^{-y}[x+y]_q^n d\mu_q(y) \\ &= \sum_{a=0}^{f-1}\chi(a)\lim_{N\to\infty}\frac{1}{[fp^N]_q}\sum_{y=0}^{fp^N-1}[a+x+fy]_q^n \\ &= \frac{1}{(1-q)^n}\sum_{a=0}^{f-1}\chi(a)\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{l(x+a)}\frac{l}{[lf]_q} \\ &= \sum_{a=0}^{f-1}\chi(a)\sum_{m=0}^{\infty}\left(-n[x+a+mf]_q^{n-1}\right) \\ &= -n\sum_{m=0}^{\infty}\chi(m)[x+m]_q^{n-1}.\end{aligned} \tag{8}$$

By (7) and (8), we can give the generating function for the generalized q-Bernoulli polynomials attached to χ as follows:

$$F_{\chi,q}(x,t) = -t\sum_{m=0}^{\infty}\chi(m)e^{[x+m]_q t} = \sum_{n=0}^{\infty}\beta_{n,\chi,q}(x)\frac{t^n}{n!}. \tag{9}$$

From (1), (8) and (9), we note that

$$\begin{aligned}\beta_{n,\chi,q}(x) &= \frac{1}{[f]_q}\sum_{a=0}^{f-1}\chi(a)\int_{\mathbb{Z}_p} q^{-fy}[a+x+fy]_q^n d\mu_{q^f}(y)\\ &= [f]_q^{n-1}\sum_{a=0}^{f-1}\chi(a)\beta_{n,q^f}\left(\frac{a+x}{f}\right).\end{aligned}$$

In the special case, $x=0$, the sequence $\beta_{n,\chi,q}(0)=\beta_{n,\chi,q}$ are called the n-th generalized q-Bernoulli numbers attaches to χ.

Let us consider the higher-order q-Bernoulli polynomials attached to χ as follows:

$$\underbrace{\int_X\cdots\int_X}_{r\text{ times}}\left(\prod_{i=1}^r\chi(x_i)\right)e^{[x+x_1+\cdots+x_r]_q t}q^{-(x_1+\cdots+x_r)}d\mu_q(x_1)\cdots d\mu_q(x_r)=\sum_{n=0}^{\infty}\beta_{n,\chi,q}^{(r)}(x)\frac{t^n}{n!}, \tag{10}$$

where $\beta_{n,\chi,q}^{(r)}(x)$ are called the n-th generalized q-Bernoulli polynomials of order r attaches to χ.

By (10), we see that

$$\begin{aligned}\beta_{n,\chi,q}^{(r)}(x) &= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-q^x)^l\sum_{a_1,\cdots,a_r=0}^{f-1}\left(\prod_{i=1}^r\chi(a_i)\right)q^{l\sum_{i=1}^r a_i}\frac{l^r}{[lf]_q^r}\\ &= [f]_q^{n-r}\sum_{a_1,\cdots,a_r=0}^{f-1}\left(\prod_{i=1}^r\chi(a_i)\right)\beta_{n,q^f}^{(r)}\left(\frac{x+a_1+\cdots+a_r}{f}\right).\end{aligned} \tag{11}$$

In the special case, $x=0$, the sequence $\beta_{n,\chi,q}^{(r)}(0)=\beta_{n,\chi,q}^{(r)}$ are called the n-th generalized q-Bernoulli numbers of order r attaches to χ.

By (10) and (11), we obtain the following theorem.

Theorem 3. Let χ be the primitive Dirichlet's character with conductor $f\in\mathbb{N}$. For $n\in\mathbb{Z}_+, r\in\mathbb{N}$, we have

$$\begin{aligned}\beta_{n,\chi,q}^{(r)}(x) &= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-q^x)^l\sum_{a_1,\cdots,a_r=0}^{f-1}\left(\prod_{i=1}^r\chi(a_i)\right)q^{l\sum_{i=1}^r a_i}\frac{l^r}{[lf]_q^r}\\ &= [f]_q^{n-r}\sum_{a_1,\cdots,a_r=0}^{f-1}\left(\prod_{i=1}^r\chi(a_i)\right)\beta_{n,q^f}^{(r)}\left(\frac{x+a_1+\cdots+a_r}{f}\right).\end{aligned}$$

For $h\in\mathbb{Z}$, and $r\in\mathbb{N}$, we introduce the extended higher-order q-Bernoulli polynomials as follows:

$$\beta_{n,q}^{(h,r)}(x)=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}q^{\sum_{j=1}^r(h-j-1)x_j}[x+x_1+\cdots+x_r]_q^n d\mu_q(x_1)\cdots d\mu_q(x_r). \tag{12}$$

From (12), we note that

$$\beta_{n,q}^{(h,r)}(x)=\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{\binom{l+h-1}{r}}{\binom{l+h-1}{r}_q}\frac{r!}{[r]_q!}, \tag{13}$$

and

$$\beta_{n,q}^{(h,r)}(x) = [f]_q^{n-r} \sum_{a_1,\cdots,a_r=0}^{f-1} q^{\sum_{j=1}^{r}(h-j)a_j} \beta_{n,q^f}^{(h,r)}\left(\frac{x+a_1+\cdots+a_r}{f}\right).$$

In the special case, $x=0$, $\beta_{n,q}^{(h,r)}(0) = \beta_{n,q}^{(h,r)}$ are called the n-th (h,q)-Bernoulli numbers of order r.

By (13), we obtain the following theorem.

Theorem 4. For $h \in \mathbb{Z}, r \in \mathbb{N}$, we have

$$\beta_{n,q}^{(h,r)}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-q^x)^l \frac{\binom{l+h-1}{r}}{\binom{l+h-1}{r}_q} \frac{r!}{[r]_q!},$$

and

$$\beta_{n,q}^{(h,r)}(x) = [f]_q^{n-r} \sum_{a_1,\cdots,a_r=0}^{f-1} q^{\sum_{j=1}^{r}(h-j)a_j} \beta_{n,q^f}^{(h,r)}\left(\frac{x+a_1+\cdots+a_r}{f}\right).$$

Let χ be the primitive Dirichlet's character with conductor $f \in \mathbb{N}$. Then we consider the generalized (h,q)-Bernoulli polynomials attached to χ of order r as follows:

$$\beta_{n,\chi,q}^{(h,r)}(x) = \underbrace{\int_X \cdots \int_X}_{r \text{ times}} q^{\sum_{j=1}^{r}(h-j-1)x_j} \left(\prod_{j=1}^{r}\chi(x_j)\right) [x+x_1+\cdots+x_r]_q^n d\mu_q(x_1)\cdots d\mu_q(x_r). \quad (14)$$

By (14), we see that

$$\beta_{n,\chi,q}^{(h,r)}(x) = [f]_q^{n-r} \sum_{a_1,\cdots,a_r=0}^{f-1} q^{\sum_{j=1}^{r}(h-j)a_j} \left(\prod_{j=1}^{r}\chi(a_j)\right) \beta_{n,q^f}^{(h,r)}\left(\frac{x+a_1+\cdots+a_r}{f}\right). \quad (15)$$

In the special case, $x=0$, $\beta_{n,\chi,q}^{(h,r)}(0) = \beta_{n,\chi,q}^{(h,r)}$ are called the n-th generalized (h,q)-Bernoulli numbers attached to χ of order r.

From (14) and (15), we note that

$$\beta_{n,\chi,q}^{(h,r)} = (q-1)\beta_{n+1,\chi,q}^{(h-1,r)} + \beta_{n,\chi,q}^{(h-1,r)}.$$

By (12), it is easy to show that

$$\begin{aligned}\beta_{n,\chi,q}^{(h,r)} &= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [x_1+\cdots+x_r]_q^n q^{\sum_{j=1}^{r}(h-j-1)x_j} d\mu_q(x_1)\cdots d\mu_q(x_r) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1+\cdots+x_r]_q^n \{[x_1+\cdots+x_r]_q(q-1)+1\} q^{\sum_{j=1}^{r}(h-j-2)x_j} d\mu_q(x_1)\cdots d\mu_q(x_r).\end{aligned} \quad (16)$$

Thus, we have

$$\beta_{n,q}^{(h,r)} = (q-1)\beta_{n+1,q}^{(h-1,r)} + \beta_{n,q}^{(h-1,r)}.$$

From (12) and (16), we can also derive

$$
\begin{aligned}
&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r \text{ times}} q^{(n-2)x_1+(n-3)x_2+\cdots+(n-r-1)x_r} d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_r)} q^{n(x_1+\cdots+x_r)} q^{-x_1-2x_2-\cdots-rx_r} d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\sum_{l=0}^{n}\binom{n}{l}(q-1)^l \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_r]_q^l q^{-(x_1+\cdots+x_r)} q^{-x_1-2x_2-\cdots-rx_r} d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\sum_{l=0}^{n}\binom{n}{l}(q-1)^l \beta_{l,q}^{(0,r)},
\end{aligned} \tag{17}
$$

and

$$
\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{(n-2)x_1+(n-3)x_2+\cdots+(n-r-1)x_r} d\mu_q(x_1)\cdots d\mu_q(x_r)=\frac{\binom{n-1}{r}}{\binom{n-1}{r}_q}\frac{r!}{[r]_q!}. \tag{18}
$$

It is easy to see that

$$
\sum_{j=0}^{n}\binom{n}{j}(q-1)^j\int_{\mathbb{Z}_p}[x]_q^j q^{(h-2)x} d\mu_q(x)=\int_{\mathbb{Z}_p}((q-1)[x]_q+1)^n q^{(h-2)x} d\mu_q(x)=\frac{n+h-1}{[n+h-1]_q}. \tag{19}
$$

By (16), (17), (18) and (19), we obtain the following theorem.

Theorem 5. For $h\in\mathbb{Z}, r\in\mathbb{N}$ and $n\in\mathbb{Z}_+$, we have

$$
\beta_{n,q}^{(h,r)}=(q-1)\beta_{n+1,q}^{(h-1,r)}+\beta_{n,q}^{(h-1,r)},
$$

and

$$
\sum_{l=0}^{n}\binom{n}{l}(q-1)^l\beta_{l,q}^{(0,r)}=\frac{\binom{n-1}{r}}{\binom{n-1}{r}_q}\frac{r!}{[r]_q!}.
$$

Furthermore, we get

$$
\sum_{l=0}^{n}\binom{n}{l}(q-1)^l\beta_{l,q}^{(h,1)}=\frac{n+h-1}{[n+h-1]_q}.
$$

Now, we consider the polynomials of $\beta_{n,q}^{(0,r)}(x)$ by

$$
\begin{aligned}
\beta_{n,q}^{(0,r)}(x)&=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r \text{ times}}[x+x_1+\cdots+x_r]_q^n q^{-2x_1-3x_2-\cdots-(r-1)x_r} d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{\binom{l-1}{r}}{\binom{l-1}{r}_q}\frac{r!}{[r]_q!}.
\end{aligned} \tag{20}
$$

By (20), we obtain the following theorem.

Theorem 6. For $r\in\mathbb{N}$ and $n\in\mathbb{Z}_+$, we have

$$
(1-q)^n\beta_{n,q}^{(0,r)}(x)=\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{\binom{l-1}{r}}{\binom{l-1}{r}_q}\frac{r!}{[r]_q!}.
$$

By using multivariate p-adic q-integral on $\mathbb{Z}_p$, we see that

$$\begin{aligned}
& q^{nx}\frac{\binom{n-1}{r}}{\binom{n-1}{r}_q}\frac{r!}{[r]_q!} \\
&= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}} q^{nx+(n-2)x_1+\cdots+(n-r-1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left((q-1)[x+x_1+\cdots+x_r]_q+1\right)^n q^{-2x_1\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= \sum_{l=0}^{n}\binom{n}{l}(q-1)^l\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_q^l q^{-2x_1\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= \sum_{l=0}^{n}\binom{n}{l}(q-1)^l\beta_{l,q}^{(0,r)}(x).
\end{aligned}$$

Therefore, we obtain the following corollary.

Corollary 7. For $r\in\mathbb{N}$ and $n\in\mathbb{Z}_+$, we have

$$q^{nx}\frac{\binom{n-1}{r}}{\binom{n-1}{r}_q}\frac{r!}{[r]_q!}=\sum_{l=0}^{n}\binom{n}{l}(q-1)^l\beta_{l,q}^{(0,r)}(x).$$

It is easy to show that

$$\begin{aligned}
& \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}[x+x_1+\cdots+x_r]_q^n q^{-2x_1\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= [f]_q^{n-r}\sum_{i_1,\cdots,i_r=0}^{f-1} q^{-\sum_{l=1}^{r} l i_l} \\
&\qquad \times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-f\sum_{l=1}^{r}(l+1)x_l}\left[\frac{x+\sum_{l=1}^{r} i_l}{f}+\sum_{l=1}^{r}x_l\right]_{q^f}^n d\mu_{q^f}(x_1)\cdots d\mu_{q^f}(x_r).
\end{aligned} \tag{21}$$

From (21), we note that

$$\beta_{n,q}^{(0,r)}(x)=[f]_q^{n-r}\sum_{i_1,\cdots,i_r=0}^{f-1}q^{-i_1-2i_2-\cdots-ri_r}\beta_{n,q^f}^{(0,r)}\left(\frac{x+i_1+\cdots+i_r}{f}\right).$$

From the multivariate p-adic q-integral on $\mathbb{Z}_p$, we have

$$\begin{aligned}
& \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}[x+x_1+\cdots+x_r]_q^n\, q^{-2x_1-3x_2-\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left([x]_q+q^x[x_1+\cdots+x_r]_q\right)^n q^{-2x_1-3x_2-\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= \sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{lx}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_r]_q^l q^{-2x_1-3x_2-\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r),
\end{aligned} \tag{22}$$

and

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}[x+y+x_1+\cdots+x_r]_q^n\,q^{-2x_1-3x_2-\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r)$$
$$=\sum_{l=0}^{n}\binom{n}{l}[y]_q^{n-l}q^{ly}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_q^l q^{-2x_1-3x_2-\cdots-(r+1)x_r}d\mu_q(x_1)\cdots d\mu_q(x_r). \tag{23}$$

By (22) and (23), we obtain the following corollary.

Corollary 8. For $r\in\mathbb{N}$ and $n\in\mathbb{Z}_+$, we have

$$\beta_{n,q}^{(0,r)}(x)=\sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{lx}\beta_{l,q}^{(0,r)},$$

and

$$\beta_{n,q}^{(0,r)}(x+y)=\sum_{l=0}^{n}\binom{n}{l}[y]_q^{n-l}q^{ly}\beta_{l,q}^{(0,r)}(x).$$

Now, we also consider the polynomial of $\beta_{n,q}^{(h,1)}(x)$. From the integral equation on $\mathbb{Z}_p$, we note that

$$\begin{aligned}\beta_{n,q}^{(h,1)}(x)&=\int_{\mathbb{Z}_p}[x+x_1]_q^n\,q^{x_1(h-2)}d\mu_q(x_1)\\&=\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{l+h-1}{[l+h-1]_q}.\end{aligned}\tag{24}$$

By (24), we easily get

$$\begin{aligned}\beta_{n,q}^{(h,1)}(x)&=\frac{1}{(1-q)^{n-1}}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}l}{1-q^{l+h-1}}+\frac{h-1}{(1-q)^{n-1}}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{1-q^{l+h-1}}\\&=\frac{-n}{(1-q)^{n-1}}\sum_{l=0}^{n-1}\frac{\binom{n-1}{l}(-1)^l q^{x}q^{lx}}{1-q^{l+h}}+\frac{h-1}{(1-q)^{n-1}}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{1-q^{l+h-1}}\\&=-n\sum_{m=0}^{\infty}q^{hm+x}[x+m]_q^{n-1}+(h-1)(1-q)\sum_{m=0}^{\infty}q^{(h-1)m}[x+m]_q^{n}.\end{aligned}$$

Thus, we obtain the following theorem.

Theorem 9. For $h\in\mathbb{Z}$ and $n\in\mathbb{Z}_+$, we have

$$\beta_{n,q}^{(h,1)}(x)=-n\sum_{m=0}^{\infty}q^{hm+x}[x+m]_q^{n-1}+(h-1)(1-q)\sum_{m=0}^{\infty}q^{(h-1)m}[x+m]_q^{n}.$$

From the definition of p-adic q-integral on $\mathbb{Z}_p$, we note that

$$\int_{\mathbb{Z}_p}q^{(h-2)x_1}[x+x_1]_q^n d\mu_q(x_1)=\frac{1}{[f]_q}\sum_{i=0}^{f-1}q^{(h-1)i}[i]_q^n\int_{\mathbb{Z}_p}\left[\frac{x+i}{f}+x_1\right]_{q^f}^n q^{f(h-2)x_1}d\mu_{q^f}(x_1).$$

Thus, we have

$$\beta_{n,q}^{(h,1)}(x) = \frac{1}{[f]_q} \sum_{i=0}^{f-1} q^{(h-1)i} [i]_q^n \beta_{n,q^f}^{(h,1)} \left(\frac{x+i}{f} \right).$$

By (24), we easily get

$$\int_{\mathbb{Z}_p} [x+x_1]_q^n \, q^{x_1(h-2)} d\mu_q(x_1) = q^{-x} \int_{\mathbb{Z}_p} [x+x_1]_q^n \{[x+x_1]_q(q-1)+1\} q^{x_1(h-3)} d\mu_q(x_1). \tag{25}$$

From (25), we have

$$\beta_{n,q}^{(h,1)}(x) = q^{-x} \left((q-1)\beta_{n+1,q}^{(h-1,1)}(x) + \beta_{n,q}^{(h-1,1)}(x) \right).$$

That is,

$$q^x \beta_{n,q}^{(h,1)}(x) = (q-1)\beta_{n+1,q}^{(h-1,1)}(x) + \beta_{n,q}^{(h-1,1)}(x).$$

By (24) and (25), we easily see that

$$\begin{aligned} \int_{\mathbb{Z}_p} q^{(h-2)x_1} [x+x_1]_q^n d\mu_q(x_1) &= \int_{\mathbb{Z}_p} q^{(h-2)x_1} ([x]_q + q^x [x_1]_q)^n d\mu_q(x_1) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \int_{\mathbb{Z}_p} q^{(h-2)x_1} [x_1]_q^l d\mu_q(x_1), \end{aligned} \tag{26}$$

and

$$\begin{aligned} &q^{h-1} \int_{\mathbb{Z}_p} q^{(h-2)x_1} [x_1+1+x]_q^n d\mu_q(x_1) - \int_{\mathbb{Z}_p} q^{(h-2)x_1} [x+x_1]_q^n d\mu_q(x_1) \\ &= q^x n [x]_q^{n-1} + h(q-1)[x]_q^n - (q-1)[x]_q^n. \end{aligned}$$

For $x=0$, this give

$$q^{h-1} \int_{\mathbb{Z}_p} q^{(h-2)x_1} [x_1+1]_q^n d\mu_q(x_1) - \int_{\mathbb{Z}_p} q^{(h-2)x_1} [x_1]_q^n d\mu_q(x_1) = \begin{cases} 1, & \text{if } n=1, \\ 0, & \text{if } n>1, \end{cases} \tag{27}$$

and

$$\beta_{0,q}^{(h,1)} = \int_{\mathbb{Z}_p} q^{(h-2)x_1} d\mu_q(x_1) = \frac{h-1}{[h-1]_q}.$$

From (26) and (27), we can derive the recurrence relation for $\beta_{n,q}^{(h,1)}$ as follows:

$$q^{h-1}\beta_{n,q}^{(h,1)}(1) - \beta_{n,q}^{(h,1)} = \delta_{n,1}, \tag{28}$$

where $\delta_{n,1}$ is kronecker symbol.

By (26), (27) and (28), we obtain the following theorem.

Theorem 10. For $h \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we have

$$\beta_{n,q}^{(h,1)}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \beta_{l,q}^{(h,1)},$$

and

$$q^{h-1}\beta_{n,q}^{(h,1)}(x+1) - \beta_{n,q}^{(h,1)} = q^x n [x]_q^{n-1} + h(q-1)[x]_q^n - (q-1)[x]_q^n.$$

Furthermore,

$$q^{h-2}(q-1)\beta_{n+1,q}^{(h-1,1)}(1) + q^{h-2}\beta_{n,q}^{(h-1,1)}(1) - \beta_{n,q}^{(h,1)} = \delta_{n,1},$$

where $\delta_{n,1}$ is kronecker symbol.

From the definition of p-adic q-integral on $\mathbb{Z}_p$, we note that

$$\int_{\mathbb{Z}_p} q^{-(h-2)x_1}[1-x+x_1]_{q^{-1}}^n d\mu_{q^{-1}}(x_1) = (-1)^n q^{n+h-2} \int_{\mathbb{Z}_p} q^{(h-2)x_1}[x+x_1]_q^n d\mu_q(x_1). \tag{29}$$

By (29), we see that

$$\beta_{n,q^{-1}}^{(h,1)}(1-x) = (-1)^n q^{n+h-2} \beta_{n,q}^{(h,1)}(x).$$

Note that

$$B_n(1-x) = \lim_{q\to 1} \beta_{n,q^{-1}}^{(h,1)}(1-x) = \lim_{q\to 1}(-1)^n q^{n+h-2}\beta_{n,q}^{(h,1)}(x) = (-1)^n B_n(x),$$

where $B_n(x)$ are the n-th ordinary Bernoulli polynomials.

In the special case, $x=1$, we get

$$\beta_{n,q^{-1}}^{(h,1)} = (-1)^n q^{n+h-2}\beta_{n,q}^{(h,1)}(1) = (-1)^n q^{n-1}\beta_{n,q}^{(h,1)} \text{ if } n>1.$$

It is not difficult to show that

$$[f]_q^{n-1}\sum_{l=0}^{f-1} q^{l(h-1)} \int_{\mathbb{Z}_p} \left[x+\frac{l}{f}+x_1\right]_{q^f}^n q^{f(h-2)x_1} d\mu_{q^f}(x_1) = \int_{\mathbb{Z}_p} [fx+x_1]_q^n \, q^{(h-2)x_1} d\mu_q(x_1), f\in\mathbb{N}.$$

That is,

$$[f]_q^{n-1}\sum_{l=0}^{f-1} q^{l(h-1)} \beta_{n,q^f}^{(h,1)}\left(x+\frac{l}{f}\right) = \beta_{n,q}^{(h,1)}(fx).$$

Let us consider Barnes' type multiple q-Bernoulli polynomials. For $w_1, w_2, \cdots, w_r \in \mathbb{Z}_p$, and $\delta_1, \delta_2, \cdots, \delta_r \in \mathbb{Z}$, we define Barnes' type multiple q-Bernoulli polynomials as follows:

$$\begin{aligned}
&\beta_{n,q}^{(r)}(x \mid w_1, \cdots, w_r : \delta_1, \cdots, \delta_r) \\
&= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} [w_1x_1 + \cdots + w_rx_r + x]_q^n q^{\sum_{j=1}^r (\delta_j - 1)x_j} d\mu_q(x_1)\cdots d\mu_q(x_r).
\end{aligned} \tag{30}$$

From (30), we can easily derive the following equation:

$$\begin{aligned}
&\beta_{n,q}^{(r)}(x \mid w_1, \cdots, w_r : \delta_1, \cdots, \delta_r) \\
&= \frac{1}{(1-q)^n}\sum_{l=0}^n \binom{n}{l}(-1)^l q^{lx} \frac{(lw_1+\delta_1)(lw_2+\delta_2)\cdots(lw_r+\delta_r)}{[lw_1+\delta_1]_q[lw_2+\delta_2]_q\cdots[lw_r+\delta_r]_q}.
\end{aligned}$$

Let $\delta_r = \delta_1 + r - 1$. Then we have

$$\beta_{n,q}^{(r)}(x \mid \underbrace{w_1 \cdots w_1}_{r \text{ times}} : \delta_1, \delta_1+1 \cdots, \delta_1+r-1) = \frac{1}{(1-q)^n}\sum_{l=0}^n \binom{n}{l}(-1)^l q^{lx} \frac{\binom{lw_1+\delta_1+r-1}{r}}{\binom{lw_1+\delta_1+r-1}{r}_q}\frac{r!}{[r]_q!}.$$

Therefore, we obtain the following theorem.

Theorem 11. For $w_1 \in \mathbb{Z}_p, r\in\mathbb{N}$ and $\delta_1 \in \mathbb{Z}$, we have

$$\begin{aligned}
&\beta_{n,q}^{(r)}(x \mid \underbrace{w_1 \cdots w_1}_{r \text{ times}} : \delta_1, \delta_1+1 \cdots, \delta_1+r-1) \\
&= \frac{1}{(1-q)^n}\sum_{l=0}^n \binom{n}{l}(-1)^l q^{lx} \frac{\binom{lw_1+\delta_1+r-1}{r}}{\binom{lw_1+\delta_1+r-1}{r}_q}\frac{r!}{[r]_q!}.
\end{aligned}$$

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